5. Tsunami Approaching a Coast

5.1 Tsunami Propagation in a Continental Slope Region [Note: Please check the change.]

Since a tsunami can be considered to be approximately a type of long ocean wave, its propagation velocity is given by the following formula:

$$c = \sqrt{gD} \tag{5.1}$$

If the data on sea bottom topography is taken as $D = D(x, y)$, the wave propagation ray can be obtained by Snell's law. In general, a wave propagates along a path of minimum time. We apply Snell's law (which is based on the propagation of light) to the problem of the propagation of tsunamis.

（A） Tsunami propagating in the continental slope region

We assume that the coastline is straight and that the contours of the sea bed are parallel to the coastline. Further, we assume that the sea depth changes linearly and is proportional to the distance from the coastline. We set the origin at a point on the coastline, and take the y-axis along the coastline, and the x-axis away from the shore. We assume that the depth is given as

$$D = mx \tag{5.2}$$

Further, we assume that an unknown function $y = f(x)$ represents the wave (ray) of a tsunami and satisfies the condition that the following time integral has the minimum value:

$$T(f) = \int_{A \to B} \frac{ds}{\sqrt{gD}} = \frac{1}{\sqrt{gm}} \int_{A \to B} \frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{x}} = \frac{1}{\sqrt{gm}} \int_{A \to B} \frac{\sqrt{1 + (y')^2}}{\sqrt{x}} dx \tag{5-3}$$

Finally, our problem reduces to the following.

Search for a function $y = f(x)$ such that the following integral

$$I = \int_{x_A}^{x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{x}} dx \tag{5-4}$$

becomes a minimum.

This is a mathematical problem of calculus of variations ("Henbunho") using which we can determine a function that makes the value of an integral maximum or minimum.

[Mathematical Note: Method of the calculus of variations]

If a function $y = f(x)$ maximizes (r minimizes) the value of the integral

$$I = \int_a^b F(x, y, y')dx \tag{5-5}$$

, $y = f(x)$ satisfies "Euler's equation,"

$$F_y - \frac{d}{dx}F_{y'} = 0 \tag{5-6}$$

In the special case that $F(x, y, y')$ does not contain x explicitly, Euler's equation is simply given by

$$F - y'F_{y'} = \text{Const} \tag{5-7}$$

[Proof] Differentiate (5-7) and we have

$$F_x + F_y y' + F_{y'} y'' - y'' F_{y''} - y' \frac{d}{dx} F_{y''} = y' \left(F_y - \frac{d}{dx} F_{y'} \right) = 0$$

Further, if $F(x, y, y')$ does not contain y explicitly, Euler's equation becomes simply

$$F_{y'} = \text{Const.} \tag{5-8}$$

In the present problem, we have

$$F(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{x}} \tag{5.9}$$

This does not contain y explicitly, and hence we can apply (5-8); in such a case, we have

$$\frac{y'}{\sqrt{x}\sqrt{1 + y'^2}} = C$$

Solving this for y', we set $a = 1/(2C^2)$; this equation then becomes

$y' = \pm\sqrt{\dfrac{x}{2a - x}}$ We then substitute $x = a(1 - \cos t)$; then we have

$$y' = \frac{dy}{dx} = \pm\sqrt{\frac{1 - \cos t}{1 + \cos t}} = \pm\tan\frac{t}{2}$$

and hence,

$$\frac{dy}{dt} = y'\frac{dx}{dt} = \pm\tan\frac{t}{2}a\sin t = \pm 2a\sin^2\frac{t}{2} = \pm a(1 - \cos t)$$

Integrating this wrt t, we have

$$y = a(t - \sin t)$$

We finally reach the following form.

$$\begin{cases} x = a(1 - \cos t) \\ y = a(t - \sin t) \end{cases} \tag{5.10}$$

This is the (x, y) relation expressed by the parameter t. Even if (5.10) does not have the form of an explicit function, we can consider our problem to be solved. (5.10) represents a cycloid curve. If, using a chalk, we can mark a point on the tire of a bicycle, the trajectory of the mark when the tire moves is a cycloid.

Finally, we conclude that a point on a tsunami wave in the continental slope sea region will take the form of a cycloid.

［Problem］ Solve similar problems for: (a) the depth distribution is given by $D(x) = bx^2$ and (b) $D(x) = ke^{bx}$.

[Problem] Obtain the time integral (6-3) for one cycloid $(0 \le t \le 2\pi)$.

［Edge wave］ If the tsunami wave is reflected at the coast and draws cycloids in the continental slope sea region, then we can understand that there is a type of "wave" that moves parallel to the coastline. We call such a wave "edge wave."

[Problem] Prove that the phase speed of an edge wave is a half that of a long wave corresponding to the depth at the deepest point of the cycloid.

(B) Tsunami wave trapped in a skirt sea region around an isolated island

Next, we consider the case of tsunamis moving around an isolated island. The area of the island is small and the sea bottom topography is cylindrically symmetric; the depth is given by

$$D(r) = br^2 \tag{5.11}$$

In a polar coordinate system, an element of the line segment is given by $ds = \sqrt{r^2 + r'^2}\, d\theta$; hence, the time integral is given by

$$T = \int \frac{ds}{\sqrt{gD}} = \frac{1}{\sqrt{gb}} \int \frac{\sqrt{r^2 + r'^2}}{r} d\theta \tag{5-12}$$

In this case, the integrated functional becomes $F(\theta, r, r') = \sqrt{r^2 + r'^2}/r$, which is a case where the independent variable θ is not contained explicitly.（Note： Whether the letters x, y or θ, r are used is unimportant)

Euler's equation is

$$F - r'F_{r'} = r\Big/\sqrt{r^2 + r'^2} = C$$

Solving this for r', we have

$r' = \pm\sqrt{\dfrac{1-C^2}{C^2}}r$ If we use K for the square root term,

$\dfrac{dr}{d\theta} = Kr$ This is easily solved as

$$r = Ae^{K\theta} \quad (5\text{-}13)$$

(5-13) shows a logarithmic spiral, which we observe in nature, for example, in the shell of a snail; it is a somewhat "self-similar shape."

In this manner, we reach the conclusion that

Once a tsunami reaches the skirt sea area around an isolated island, then all its energy is transferred to the coast of the isolated island. Therefore, a tsunami wave is tends to become amplified in the sea area around an isolated island with a large skirt sea area.

5.2 Amplification of a tsunami wave

Let us consider the amplification of a tsunami wave moving between two waves (rays). At the origin, the width of the channel between the rays is b_0, and the wave crest is situated at A_0B_0; here, the depth is D_0, and the tsunami wave has an amplitude of a_0. This wave crest approaches the shoreline A_SB_S, and then the amplitude is assumed to be a_S. The wave energy ΔE passing at A_0B_0 in the time interval Δt is given by

$$\begin{aligned}\Delta E &= (T+V)/L \times c_{G0} \times b_0 \times \Delta t = \rho g a_0 b_0 c_{G0}\Delta t/2 \\ &= \rho g a_0^2 b_0(1+2kD_0 \cos ech2kD_0)c_0\Delta t/4\end{aligned} \quad (5\text{-}14)$$

Further, the energy at a shallow place A_SB_S is given by

$$\Delta E = \rho g a_S^2 b_S(1+2kD_S \cos ech2kD_S)c_S\Delta t/4 \quad (5\text{-}15)$$

If we assume that there is no energy dissipation on the way, (5-14) and (5-15) assume identical values; hence, we have the following amplification formula.

$$\frac{a_S}{a_0} = \sqrt{\frac{b_0(1+2kD_0 \cos ech2kD_0)\sqrt{\tanh kD_0}}{b_S(1+2kD_S \cos ech2kD_S)\sqrt{\tanh kD_S}}} \quad (5\text{-}16)$$

Since a tsunami wave can be approximated as a long wave, we can substitute $kD_0 \to 0, kD_S \to 0, \tanh kD_0 \to kD_0, \tanh kD_S \to kD_S$; the amplification ratio is simply given by

$$\frac{a_S}{a_0} = \left(\frac{b_S}{b_0}\right)^{-1/2} \left(\frac{D_S}{D_0}\right)^{-1/4} \qquad (5\text{-}17)$$

Thus we have proved that the height of a tsunami wave changes in inverse proportion to the four power square root of the depth ratio (Green's Law). This formula is applicable up to Michell's wave-breaking limit:

$$a_S \le 0.73 D_S \qquad (5\text{-}18)$$

Actually, wave breaking begins at $a_S \le 0.5 D_S$, and the energy conservation law will not be applicable; it has the applicable limit given by (5-17).

［Example］ If the amplitude of a tsunami wave at a depth of 4000m is 1 m, then its height at 10m depth becomes 4.47m.